Period 3, 8/19/24

e) $\lim_{x \to c} f(x)$ exists at every c in the interval $(-1, 1)$

$(-1,1) \Rightarrow -1 < x < 1$

$[-1,1] \Rightarrow -1 \leq x \leq 1$

$\lim_{x \to 1} F(x) = DNE$

$\left.\begin{array}{l} \lim_{x \to 1^+} F(x) = 1 \\ \lim_{x \to 1^-} F(x) = 2 \end{array}\right\}$ Not Same

a) $\lim_{x \to -1^+} f(x) = 1$

f) $\lim_{x \to 1^+} f(x) = 1$

b) $\lim_{x \to 2} f(x) = DNE$

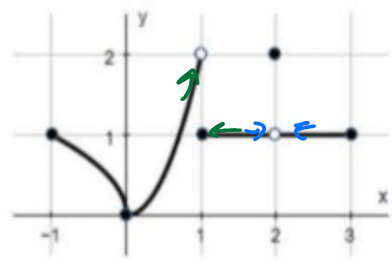
g) $\lim_{x \to 1} f(x) = DNE$

c) $\lim_{x \to 2} f(x) = 2$

h) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = 0$

d) $\lim_{x \to 1^-} f(x) = 2$

i) $\lim_{x \to c} f(x)$ exists at every c in the interval $(1, 3)$



$\lim_{x \to 2} F(x) = 1$

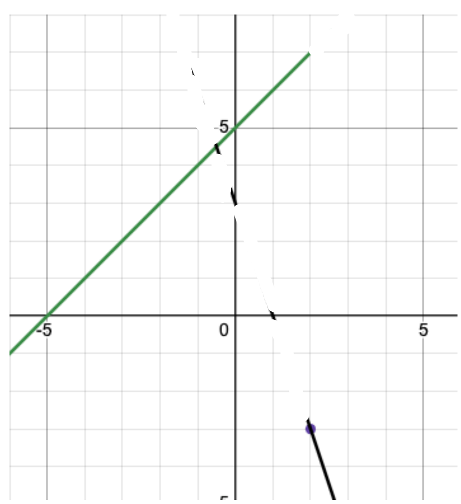
$F(2) = 2$

3. (a) Graph the following: $f(x) = \begin{cases} x+5 & x < 2 \\ -3 & x = 2 \\ -3x+3 & x > 2 \end{cases}$

(b) Use your graph to find $\lim_{x \to 2} f(x)$. If the limit does not exist, explain why.

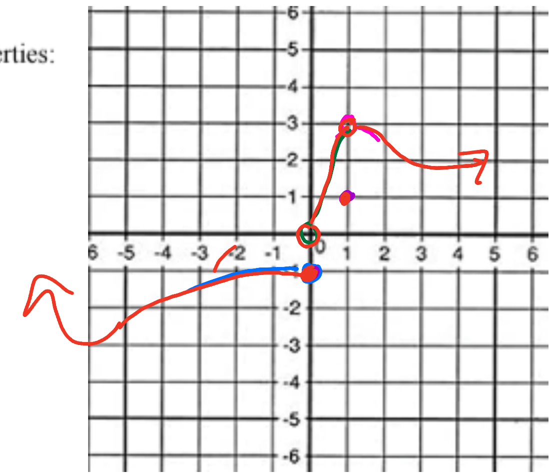
$\left.\begin{array}{l} \lim_{x \to 2^+} F(x) = -3 \\ \lim_{x \to 2^-} F(x) = 7 \end{array}\right\}$ Not The Same

$\lim_{x \to 2} F(x) = DNE$



4. Sketch a graph of a function with the given properties:

- ☐ $\lim_{x \to 1} g(x) = 3$
- ☐ $\lim_{x \to 0^-} g(x) = -1$
- ☐ $\lim_{x \to 0^+} g(x) = 0$
- ☐ $g(0) = -1$
- ☐ $g(1) = 1$



i. $\lim_{x \to -\infty} f(x) = -1$

ii. $\lim_{x \to \infty} f(x) = 2$

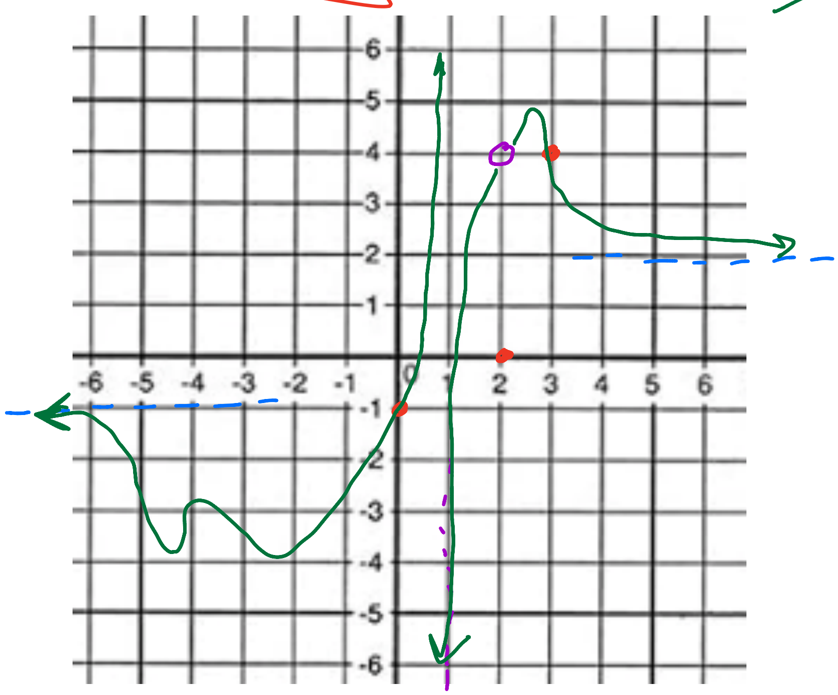
iii. $\lim_{x \to 1^+} f(x) = -\infty$

iv. $\lim_{x \to 2^-} f(x) = 4$

v. $f(2) = 0$

vi. $f(0) = -1$

vii. $f(3) = 4$



x

$Y = 5$

1. $\lim\limits_{x \to 4} 5 = 5$

$Y = x$

2. $\lim\limits_{x \to 4} x = 4$

$\lim\limits_{x \to -6} x = -6$

3. $\lim\limits_{x \to 3} x^4 = 3^4 = 81$

THEOREM PROPERTIES OF LIMITS (Composition)

$$\lim_{x \to c} f(g(x)) = \boldsymbol{f}(\lim_{x \to c} \boldsymbol{g}(\boldsymbol{x})) = \boldsymbol{f}(\boldsymbol{L})$$

ONLY IF TWO Statements are true:

1. If $\lim\limits_{x \to c} g(x) = L$ AND
2. $f(x)$ is continuous at $x = L$

If the inner limit Does Not Exist or $f(x)$ is not continuous at $x = L$, then we have a nontraditional composition function, and you must investigate the left and right limits of $g(x)$ AND f(x), starting with the inner function.

Here is a trick:

1) $\lim\limits_{x \to c} f(g(x))$

2) Analyze how $g(x)$ is approaching $k^?$ as $x \to c$
   - *Below: use* k^-
   - *Above: use* k^+
   - *Both Sides: use* k

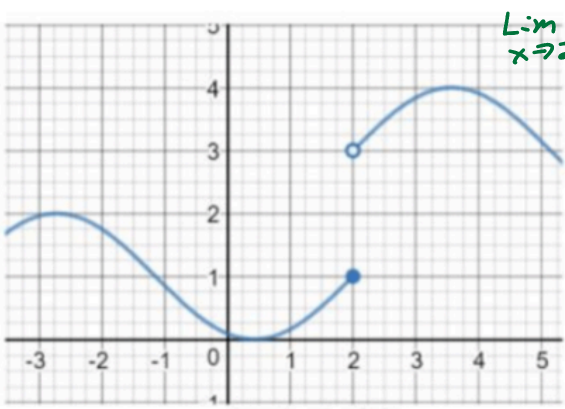
3) Then find $\lim\limits_{x \to k^?} f(x)$

$\lim\limits_{x \to c} \boldsymbol{g}(\boldsymbol{x}) = k^?$

1. The graphs of $g(x)$ and $f(x)$ are below. Find $\lim\limits_{x \to -1} g(f(x))$.

$\lim\limits_{x \to -1} F(x) = 2$

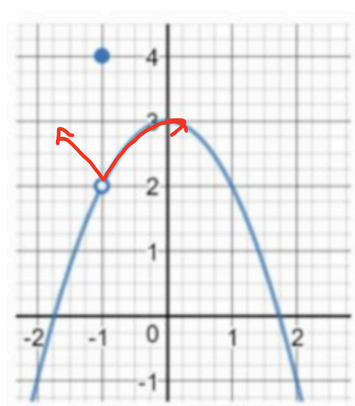
$\lim\limits_{x \to 2} g(x) = \emptyset$



Graph of $g(x)$

$\lim\limits_{x \to -1} g(F(x))$

$\lim\limits_{x \to 2^+} g(x)$



Graph of $f(x)$

Example 4.5

Find $\lim_{x \to 2} f(x)$, if it exist.

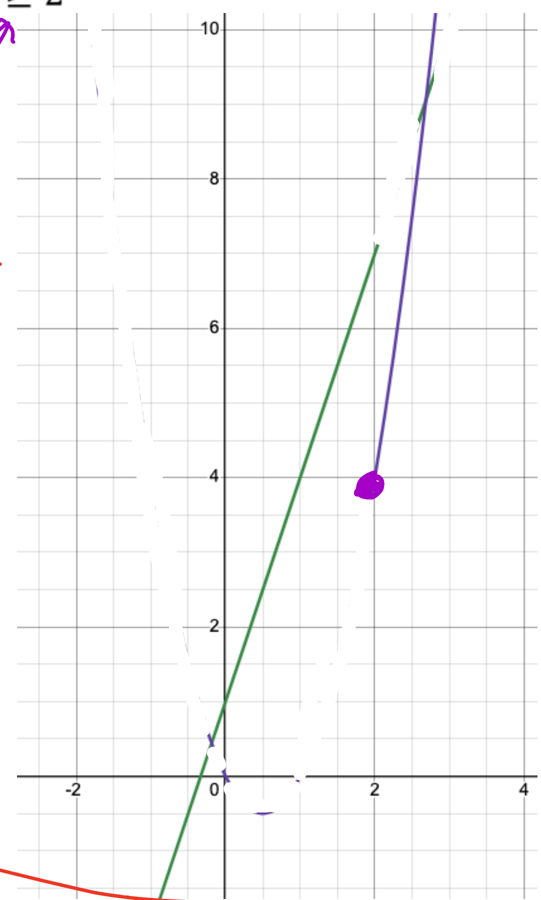
$$f(x) = \begin{cases} 3x + 1 & if \quad x < 2 \\ 2x(x-1) & if \quad x \geq 2 \end{cases}$$

$\lim_{x \to 2} F(x) = \emptyset$

$\lim_{x \to 2^-} (3x+1) = 6+1=7$

NoT The Same

$\lim_{x \to 2^+} 2x(x-1) = 2\cdot 2(2-1) = 4$



Example 6: Evaluate the following limits.

a) $\lim_{x \to 1}(-x^2 + 1) = -(1)^2+1$

$-1+1=0$

b) $\lim_{x \to 3} \frac{\sqrt{x+1}}{x-4} = \frac{\sqrt{3+1}}{3-4} = \frac{\sqrt{4}}{-1} = \frac{2}{-1}$

-2

c) $\lim_{h \to 0}(3h^2 + 2h)$

$3(0)^2+2(0)=0$

d) $\lim_{h \to 0}(3x^2 - 2xh + 5h)$

$3x^2 - 2x\cdot 0 + 5\cdot 0$

$3x^2$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{x^3-1^3}{x-1}=\lim_{x\to 1}\frac{\cancel{(x-1)}(x^2+1\cdot x+1^2)}{\cancel{(x-1)}}=1^2+1\cdot 1+1^2=3$$

Example 8

$$\lim_{x\to 0}\frac{\frac{1}{x+1}-1}{x}=\lim_{x\to 0}\frac{-1}{x+1}=\frac{-1}{1}=-1$$

$$\frac{\frac{1}{x+1}-1}{x}=\frac{\frac{1}{x+1}-\frac{x+1}{x+1}}{x}=\frac{\frac{\cancel{1}-x\cancel{-1}}{x+1}}{x}=\frac{\frac{-x}{x+1}}{\frac{x}{1}}=\frac{-\cancel{x}}{(x+1)}\cdot\frac{1}{\cancel{x}}=\frac{-1}{x+1}$$

Example 9

Find the limit: $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}-1}{x}$. $=\lim\limits_{x\to 0}\dfrac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)}=\lim\limits_{x\to 0}\dfrac{1}{\sqrt{x+1}+1}=\dfrac{1}{2}$

$$\frac{(\sqrt{x+1}-1)}{x}\cdot\frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)}=\frac{x+\cancel{1}+\cancel{1\sqrt{x+1}}-\cancel{1\sqrt{x+1}}\cancel{-1}}{x(\sqrt{x+1}+1)}$$

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}, \qquad a \neq b \Rightarrow \text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 10

a) Find the average rate of change of $g(x) = x^2 + 3x$ from 2 to x, x ≠ 2
b) Find the limit of part a as x goes to 2.

$g(2) = 2^2 + 3(2) = 10$

$g(x) = x^2 + 3x$

$$\frac{g(x) - g(2)}{x - 2} = \frac{x^2 + 3x - 10}{x - 2}$$

$$\lim_{x \to 2} \frac{g(x) - g(2)}{x - 2} = \lim_{x \to 2} \frac{(x+5)}{1} = 7 \qquad \frac{\cancel{(x-2)}(x+5)}{\cancel{x-2}}$$

The difference quotient of a function f at x is

$$(x+h)^2 = (x+h)(x+h) = x^2 + 2xh + h^2$$

$$x^2 + xh + xh + h^2$$

$$\frac{f(x+h) - f(x)}{h}, \qquad h \neq 0$$

Example 11

a) For $F(x) = 2x^2 - 3x + 1$, find the difference quotient.
b) Find the limit of part a as h approaches 0.

$F(x) = 2x^2 - 3x + 1$

$F(x+h) = 2(x+h)^2 - 3(x+h) + 1$

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) - 3(x+h) + 1 - [2x^2 - 3x + 1]}{h}$$

$$\lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{- 3x} - 3h \cancel{+ 1} \cancel{- 2x^2} \cancel{+ 3x} \cancel{- 1}}{h}$$

$$\lim_{h \to 0} \frac{4xh + 2h^2 - 3h}{h} = \lim_{h \to 0} \frac{\cancel{h}(4x + 2h - 3)}{\cancel{h}} = 4x - 2(0) - 3 = 4x - 3$$

12) $\lim_{x\to 0} \frac{\sin 4x}{x} =$

$\sin 4(\frac{\pi}{4}) \neq 4 \sin\frac{\pi}{4}$

$\sin \pi \neq 4 \cdot \frac{\sqrt{2}}{2}$

$0 \neq 2\sqrt{2}$

$\lim_{x\to 0} \frac{\sin 4x \cdot 4}{x \cdot 4}$

$\lim_{x\to 0} \frac{4 \cdot \sin 4x}{4x} = 4 \cdot 1 = 4$

13) $\lim_{\theta\to 0} \frac{\tan 6\theta}{\theta} =$

$\lim_{\theta\to 0} \frac{\frac{\sin 6\theta}{\cos 6\theta}}{\frac{\theta}{1}} =$

$\lim_{\theta\to 0} \frac{\sin 6\theta}{\cos 6\theta} \cdot \frac{1}{\theta}$

$\lim_{\theta\to 0} \frac{6 \cdot \sin 6\theta}{6\theta \cos 6\theta}$

$\lim_{\theta\to 0} \boxed{\frac{\sin 6\theta}{6\theta}} \cdot \frac{6}{\cos 6\theta}$

$1 \cdot \frac{6}{1} = 6$

$\lim_{a\to 0} \frac{\sin a}{a} = 1$

$a = 4x$

$\lim_{x\to 0} \frac{\sin 4x}{4x} = 1$

$\lim_{x\to 0} \frac{4 \cdot \sin 4x}{4 \cdot x}$

$\lim_{x\to 0} 4 \cdot \boxed{\frac{\sin 4x}{4x}} = 4 \cdot 1 = 4$

Example 14A

$\lim_{x\to} \frac{1-\cos x}{x} = 0$

$$\lim_{x\to 0} \frac{x^2}{1-\cos x} = \lim_{x\to 0} \frac{x^2}{\sin^2 x}(1+\cos x) = 1 \cdot (1+\cos 0) = 1(1+1) = 2$$

$$\frac{x^2}{(1-\cos x)} \frac{(1+\cos x)}{(1+\cos x)} = \frac{x^2(1+\cos x)}{1+\cancel{\cos x}-\cancel{\cos x}-\cos^2 x} = \frac{x^2(1+\cos x)}{\sin^2 x + \cancel{\cos^2 x} - \cancel{\cos^2 x}}$$

$\sin^2 x + \cos^2 x = 1$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 = \lim_{x \to 0} \frac{x}{\sin x}$$

Example 14

$$\lim_{x \to 0} \frac{\sin\left(\frac{\pi}{4} + x\right) - \sin\left(\frac{\pi}{4}\right)}{x} = \frac{\sin\left(\frac{\pi}{4}\right)\cos x + \sin x \cdot \cos\frac{\pi}{4} - \sin\frac{\pi}{4}}{x}$$

$\sin(a+b) = \sin a \cos b + \sin b \cos a$

$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$$\frac{\frac{\sqrt{2}}{2}\cos x + \frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}}{x}$$

$$\lim_{x \to 0} \underbrace{\frac{\frac{\sqrt{2}}{2}(\cos x - 1)}{x}}_{0} + \underbrace{\frac{\frac{\sqrt{2}}{2}\sin x}{x}}_{1}$$

$$\frac{\sqrt{2}}{2} \cdot 0 + \frac{\sqrt{2}}{2} \cdot 1 = \frac{\sqrt{2}}{2}$$

$$\lim_{x \to 0} \frac{\sin x}{5x^2 + x} = \lim_{x \to 0} \frac{\sin x}{x(5x+1)} = \lim_{x \to 0} \underbrace{\boxed{\frac{\sin x}{x}}}_{=1} \cdot \frac{1}{(5x+1)}$$

$$\lim_{x\to 0} x\csc(3x) = \lim_{x\to 0} \frac{x}{1} \cdot \frac{1}{\sin 3x} = \lim_{x\to 0} \frac{x}{\sin 3x} \cdot \frac{3}{3}$$ keep going